Closed injective systems and its fundamental limit spaces

Marcio Colombo Fenille[1]

Instituto de Ciências Matemáticas e de Computação - Universidade de São Paulo - Av. Trabalhador São-Carlense, 400, Centro, Caixa Postal 668, CEP 13560-970, São Carlos, SP, Brazil.

Abstract: In this article we introduce the concept of limit space and fundamental limit space for the so-called closed injected systems of topological spaces. We present the main results on existence and uniqueness of limit spaces and several concrete examples. In the main section of the text, we show that the closed injective system can be considered as objects of a category whose morphisms are the so-called cis-morphisms. Moreover, the transition to fundamental limit space can be considered a functor from this category into category of topological spaces. Later, we show results about properties on functors and counter-functors for inductive closed injective system and fundamental limit spaces. We finish with the presentation of some results of characterization of fundamental limite space for some special systems and the study of so-called perfect properties.

Key words: Closed injective system, fundamental limit space, category, functoriality.
Mathematics Subject Classification: 18A05, 18A30, 18B30, 54A20, 54B17.

1 Introduction

Our purpose is to introduce and study what we call category of closed injective systems and cis-morphisms, beyond the limit spaces of such systems.

We start by defining the so-called closed injective systems (CIS to shorten), and the concepts of limit space for such systems. We have particular interest in a special type of limit space, those we call fundamental limit space. Section 3 is devoted to introduce this concept and demonstrate theorems of existence and uniqueness of fundamental limit spaces. The following section, in turn, is devoted to presenting some very illustrative examples.

Section 5 is one of the most important and interesting for us. There we show that a closed injective system can be considered as object of a category, whose morphisms are the so-called cis-morphisms, which we define in this occasion. Furthermore, we prove that this category is complete with respect to direct limits, that is, all inductive system of CIS's and cis-morphisms has a direct limit.

In Section 6, we prove that the transition to the fundamental limit can be considered as a functor from category of CIS's and cis-morphisms into category of topological spaces and continuous maps.

In Section 7, we show that the transition to the direct limit in the category of CIS's and cis-morphisms is compatible (in a way) to transition to the fundamental limit space.

In section 8, we study a class of special CIS's called inductive closed injective systems. In the two following sections, we study the action of functors and counter-functors, respectively, in such systems, and present some simple applications of the results demonstrated.

We finish with the presentation of some results of characterization of fundamental limite space for some special systems, the so-called finitely-semicomponible and stationary systems, and the study of so-called perfect properties over topological spaces of a system and over its fundamental limit spaces.

[1]E-mail:fenille@icmc.usp.br

2 Closed injective system and limit spaces

Let $\{X_i\}_{i=0}^{\infty}$ be a countable collection of nonempty topological spaces. For each $i \in \mathbb{N}$, let Y_i be a closed subspace of X_i. Assume, for each $i \in \mathbb{N}$, there is a closed injective continuous map

$$f_i : Y_i \to X_{i+1}.$$

This structure is called **closed injective system**, or CIS, to shorten. We write $\{X_i, Y_i, f_i\}$ to represent this system. Moreover, by injection we mean a injective continuous map.

We say that two injection f_i and f_{i+1} are **semicomponible** if $f_i(Y_i) \cap Y_{i+1} \neq \emptyset$. In this case, we can define a new injection

$$f_{i,i+1} : f_i^{-1}(Y_{i+1}) \to X_{i+2}$$

by $f_{i,i+1}(y) = (f_{i+1} \circ f_i)(y)$, for all $y \in f_i^{-1}(Y_{i+1})$.

For convenience, we put $f_{i,i} = f_i$. Moreover, we say that f_i is always semicomponible with itself. Also, we write $f_{i,i-1}$ to be the natural inclusion of Y_i into X_i, for all $i \in \mathbb{N}$.

Given $i, j \in \mathbb{N}$, $j > i+1$, we say that f_i and f_j are **semicomponible** if $f_{i,k}$ and f_{k+1} are semicomponible for all $i+1 \leq k \leq j-1$, where

$$f_{i,k} : f_{i,k-1}^{-1}(Y_k) \to X_{k+1}$$

is defined inductively. To facilitate the notations, se f_i and f_j are semicomponible, we write

$$Y_{i,j} = f_{i,j-1}^{-1}(Y_j),$$

that is, $Y_{i,j}$ is the domain of the injection $f_{i,j}$. According to the agreement $f_{i,i} = f_i$, we have $Y_{i,i} = Y_i$.

Lemma 2.1. *If f_i and f_j are semicomponible, $i < j$, then f_k and f_l are semicomponible, for any integers k, l with $i \leq k \leq l \leq j$.*

Lemma 2.2. *If f_i and f_j are not semicomponible, then f_i and f_k are not semicomponible, for any integers $k > j$.*

Lemma 2.3. *Assume that f_i and f_j are semicomponible, with $i < j$. Then we have*

$$Y_{i,j} = (f_{j-1} \circ \cdots \circ f_i)^{-1}(Y_j) \quad \text{and} \quad f_{i,j}(Y_{i,j}) = (f_j \circ f_{i,j-1})(Y_{i,j-1}).$$

The proofs of above results are omitted.

Henceforth, since products of maps do not appear in this paper, we can sometimes omit the symbol $\circ$ in the composition of maps.

Definition 2.4. *Let $\{X_i, Y_i, f_i\}$ be a CIS. A* **limit space** *for this system is a topological space X and a collection of continuous maps $\phi_i : X_i \to X$ satisfying the following conditions:*

L.1. $X = \bigcup_{i=0}^{\infty} \phi_i(X_i)$;

L.2. Each $\phi_i : X_i \to X$ is a imbedding;

L.3. $\phi_i(X_i) \cap \phi_j(X_j) \doteq \phi_j f_{i,j-1}(Y_{i,j-1})$ if $i < j$ and f_i and f_j are semicomponible;

L.4. $\phi_i(X_i) \cap \phi_j(X_j) = \emptyset$ if f_i and f_j are not semicomponible;

where $\doteq$ indicates, besides the equality of sets, the following: If $x \in \phi_i(X_i) \cap \phi_j(X_j)$, say $x = \phi_i(x_i) = \phi_j(x_j)$, with $x_i \in X_i$ and $x_j \in X_j$, then we have necessarily $x_i \in Y_{i,j-1}$ and $x_j = f_{i,j-1}(x_i)$.

Remark 2.5. *The "pointwise identity" indicated by* $\doteq$ L.3 *reduced to identity of sets indicates only that*

$$\phi_i(X_i) \cap \phi_j(X_j) = \phi_i(Y_{i,j-1}) \cap \phi_j f_{i,j-1}(Y_{i,j-1}).$$

The existence of different interpretations of the condition L.3 is very important. Furthermore, equivalent conditions to those of the definition can be very useful. The next results give us some practical interpretations and equivalences.

Lemma 2.6. *Let $\{X, \phi_i\}$ be a limit space for the CIS $\{X_i, Y_i, f_i\}$ and suppose that f_i and f_j are semicomponible, with $i < j$. Then $\phi_j f_{i,j-1}(y_i) = \phi_i(y_i)$, for all $y_i \in Y_{i,j-1}$.*

Proof. Let $y_i \in Y_{i,j-1}$ be an arbitrary point. By condition L.3 we have $\phi_j f_{i,j-1}(y_i) \in \phi_i(X_i)$, that is, $\phi_j f_{i,j-1}(y_i) = \phi_i(x_i)$ for some $x_i \in X_i$. Again, by condition L.3, $x_i \in Y_{i,j-1}$ and $f_{i,j-1}(x_i) = f_{i,j-1}(y_i)$. Since each f_k is injective, $f_{i,j-1}$ is injective, too. Therefore $x_i = y_i$, which implies $\phi_j f_{i,j-1}(y_i) = \phi_i(y_i)$. □

Lemma 2.7. *Let $\{X, \phi_i\}$ be a limit space for the CIS $\{X_i, Y_i, f_i\}$, and suppose that f_i and f_j are semicomponible, with $i < j$. Then*

$$\phi_i(X_i - Y_{i,j-1}) \cap \phi_j(X_j - f_{i,j-1}(Y_{i,j-1})) = \emptyset.$$

Proof. It is obvious that if $x \in \phi_i(X_i - Y_{i,j-1}) \cap \phi_j(X_j - f_{i,j-1}(Y_{i,j-1}))$ then $x \in \phi_i(X_i) \cap \phi_j(X_j) \doteq \phi_j f_{i,j-1}(Y_{i,j-1})$. But this is a contradiction, since ϕ_j is an imbedding, and so $\phi_j(X_j - f_{i,j-1}(Y_{i,j-1})) = \phi_j(X_j) - \phi_j f_{i,j-1}(Y_{i,j-1})$. □

Proposition 2.8. *Let $\{X_i, Y_i, f_i\}$ be an arbitrary CIS and let $\phi_i : X_i \to X$ be imbedding into a topological space $X = \cup_{i=0}^{\infty} \phi_i(X_i)$, such that:*

L.4. *$\phi_i(X_i) \cap \phi_j(X_j) = \emptyset$ always that f_i and f_j are not semicomponible;*

L.5. *$\phi_j f_{i,j-1}(y_i) = \phi_i(y_i)$ for all $y_i \in Y_{i,j-1}$, always that f_i and f_j are semicomponible, with $i < j$;*

L.6. *$\phi_i(X_i - Y_{i,j-1}) \cap \phi_j(X_j - f_{i,j-1}(Y_{i,j-1})) = \emptyset$, always that f_i and f_j are semicomponible, $i < j$.*

Then $\{X, \phi_i\}$ is a limit space for the CIS $\{X_i, Y_i, f_i\}$.

Proof. We prove that the condition L.3 is true. Suppose that f_i and f_j are semicomponible, with $i < j$. By the condition L.5, the sets $\phi_i(X_i) \cap \phi_j(X_j)$ and $\phi_j f_{i,j-1}(Y_{i,j-1})$ are nonempty. We will prove that they are pointwise equal.

Let $x \in \phi_i(X_i) \cap \phi_j(X_j)$, say $x = \phi_i(x_i) = \phi_j(x_j)$ with $x_i \in X_i$ and $x_j \in X_j$. Suppose, by contradiction, that $x_i \notin Y_{i,j-1}$. Then $\phi_i(x_i) \in \phi_i(X_i - Y_{i,j-1})$. By the condition L.6 we must have $\phi_j(x_j) = \phi_i(x_i) \notin \phi_j(X_j - f_{i,j-1}(Y_{i,j-1}))$, that is, $\phi_j(x_j) \in \phi_j f_{i,j-1}(Y_{i,j-1})$. So $x_j \in f_{i,j-1}(Y_{i,j-1})$. Thus, there is $y_i \in Y_{i,j-1}$ such that $f_{i,j-1}(y_i) = x_j$. By the condition L.5, $\phi_i(y_i) = \phi_j f_{i,j-1}(y_i) = \phi_j(x_j)$. However, $\phi_j(x_j) = \phi_i(x_i)$. It follows that $\phi_i(y_i) = \phi_i(x_i)$, and so $x_i = y_i \in Y_{i,j-1}$, which is a contradiction. Therefore $x_i \in Y_{i,j-1}$.

In order to prove the remaining, take $x \in \phi_i(X_i) \cap \phi_j(X_j)$, $x = \phi_i(y_i) = \phi_j(x_j)$, with $y_i \in Y_{i,j-1}$ and $x_j \in X_j$. We must prove that $x_j = f_{i,j-1}(y_i)$. By the condition L.5, $\phi_j f_{i,j-1}(y_i) = \phi_i(y_i) = \phi_j(x_j)$. Thus, the desired identity is obtained by injectivity.

This proves that $\phi_i(X_i) \cap \phi_j(X_j) \doteq \phi_j f_{i,j-1}(Y_{i,j-1})$ and, so, that $\{X, \phi_i\}$ is a limite space for $\{X_i, Y_i, f_i\}$. □

Corollary 2.9. *The condition* L.3 *can be replaced by both together conditions* L.5 *and* L.6.

Proof. The Lemmas 2.6 e 2.7 and Proposition 2.8 implies that. □

Theorem 2.10. *Let $\{X_i, Y_i, f_i\}$ be a CIS. Assume that $\{X, \phi_i\}$ and $\{Z, \psi_i\}$ are two limit spaces for this CIS. Then there is a unique bijection (not necessarily continuous) $\beta : X \to Z$ such that $\psi_i = \beta \circ \phi_i$, for all $i \in \mathbb{N}$.*

Proof. Define $\beta : X \to Z$ in the follow way: For each $x \in X$, we have $x = \phi_i(x_i)$, for some $x_i \in X_i$. Then, we define $\beta(x) = \psi_i(x_i)$. We have:

⋄ *β is well defined.* Let $x \in X$ be a point with $x = \phi_i(x_i) = \phi_j(x_j)$, where $x_i \in X_i$, $x_j \in X_j$ and $i < j$. Then $x \in \phi_i(X_i) \cap \phi_j(X_j) \doteq \phi_j f_{i,j-1}(Y_{i,j-1})$ and $x_j = f_{i,j-1}(x_i)$ by the condition L.3. Thus $\psi_j(x_j) = \psi_j f_{i,j-1}(x_i) = \psi_i(x_i)$, where the latter identity follows from the condition L.3.

⋄ *β is injective.* Suppose that $\beta(x) = \beta(y)$, $x, y \in X$. Consider $x = \phi_i(x_i)$ and $y = \phi_j(y_j)$, $x_i \in X_i$, $y_j \in X_j$, $i < j$ (the case where $j < i$ is symmetrical and the case where $i = j$ is trivial). Then $\psi_i(x_i) = \beta(x) = \beta(y) = \psi_j(y_j)$. It follows that $\psi_i(x_i) = \psi_j(y_j) \in \psi_i(X_i) \cap \psi_j(X_j) \doteq \psi_j f_{i,j-1}(Y_{i,j-1})$. By the condition L.3, $x_i \in Y_{i,j-1}$ and $y_j = f_{i,j-1}(x_i)$. By the condition L.5, it follows that $\phi_i(x_i) = \phi_j f_{i,j-1}(x_i) = \phi_j(y_j)$. Therefore $x = y$.

⋄ *β is surjective.* Let $z \in Z$ be an arbitrary point. Then $z = \psi_i(x_i)$ for some $x_i \in X_i$. Take $x = \phi_i(x_i)$, and we have $\beta(x) = z$.

The uniqueness is trivial. □

3 The fundamental limit space

Definition 3.1. *Let $\{X, \phi_i\}$ be a limit space for the CIS $\{X_i, Y_i, f_i\}$. We say X has the* **weak topology** *(induced by collection $\{\phi_i\}_{i \in \mathbb{N}}$) if the following sentence is true:*

$$A \subset X \text{ is closed in } X \Leftrightarrow \phi_i^{-1}(A) \text{ is closed in } X_i \text{ for all } i \in \mathbb{N}.$$

When this occurs, we say that $\{X, \phi_i\}$ is a **fundamental limit space** *for the CIS $\{X_i, Y_i, f_i\}$.*

Proposition 3.2. *Let $\{X, \phi_i\}$ be a fundamental limit space for the CIS $\{X_i, Y_i, f_i\}$. Then $\phi_i(X_i)$ is closed in X, for all $i \in \mathbb{N}$.*

Proof. We prove that $\phi_j^{-1}(\phi_i(X_i))$ is closed in X_j for any $i, j \in \mathbb{N}$. We have

$$\phi_j^{-1}(\phi_i(X_i)) = \begin{cases} X_i & \text{if} \quad i = j \\ \emptyset & \text{if} \quad i < j \text{ and } f_i \text{ and } f_j \text{ are not semicomponible} \\ \emptyset & \text{if} \quad i > j \text{ and } f_j \text{ and } f_i \text{ are not semicomponible} \\ f_{i,j-1}(Y_{i,j-1}) & \text{if} \quad i < j \text{ and } f_i \text{ and } f_j \text{ are semicomponible} \\ f_{j,i-1}(Y_{j,i-1}) & \text{if} \quad i > j \text{ and } f_j \text{ and } f_i \text{ are semicomponible} \end{cases}.$$

In the first three cases is obvious that $\phi_j^{-1}(\phi_i(X_i))$ is closed in X_j. In the fourth case we have the following: If $j = i+1$, then $f_{i,j-1}(Y_{i,j-1}) = f_i(Y_i)$, which is closed in X_{i+1}, since f_i is a closed map. For $j > i+1$, since f_i is continuous and Y_{i+1} is closed in X_{i+1}, them $Y_{i,i+1} = f_i^{-1}(Y_{i+1})$ is closed in X_i. Thus, since f_i is closed, the Lemma 2.3 shows that $f_{i,i+1}(Y_{i,i+1}) = f_{i+1} f_i(Y_{i,i}) = f_{i+1} f_i(Y_i)$, which is closed in X_{i+1}. Again by the Lemma 2.3 we have $f_{i,j-1}(Y_{i,j-1}) = f_{j-1} f_{i,j-2}(Y_{i,j-2})$. Thus, by induction it follows that $f_{i,j-1}(Y_{i,j-1})$ is closed in X_j. The fifth case is similar to the fourth. □

Corollary 3.3. *Let $\{X, \phi_i\}$ be a fundamental limit space for the CIS $\{X_i, Y_i, f_i\}$. If X is compact, then each X_i is compact.*

Proof. Each X_i is homeomorphic to closed subspace $\phi_i(X_i)$ of X. $\square$

Proposition 3.4. *Let $\{X, \phi_i\}$ and $\{Z, \psi_i\}$ be two limit spaces for the CIS $\{X_i, Y_i, f_i\}$. If $\{X, \phi_i\}$ is a fundamental limit space, then the bijection $\beta : X \to Z$ of the Theorem 2.10 is continuous.*

Proof. Let A be a closed subset of Z. We have $\beta^{-1}(A) = \cup_{i=0}^{\infty}\phi_i(\psi_i^{-1}(A))$ and $\phi_j^{-1}(\beta^{-1}(A)) = \psi_j^{-1}(A)$. Since ψ_j is continuous and X has the weak topology, we have that $\beta^{-1}(A)$ is closed in X. $\square$

Theorem 3.5. (UNIQUENESS OF THE FUNDAMENTAL LIMIT SPACE) *Let $\{X, \phi_i\}$ and $\{Z, \psi_i\}$ be two fundamental limit spaces for the CIS $\{X_i, Y_i, f_i\}$. Then, the bijection $\beta : X \to Z$ of the Theorem 2.10 is a homeomorphism. Moreover, β is the unique homeomorphism from X onto Z such that $\psi_i = \beta \circ \phi_i$, for all $i \in \mathbb{N}$.*

Proof. Let $\beta' : Z \to X$ be the inverse map of the bijection β. By preceding proposition, β and β' are both continuous maps. Therefore β is a homeomorphism. The uniqueness is the same of the Theorem 2.10. $\square$

Theorem 3.6. (EXISTENCE OF FUNDAMENTAL LIMIT SPACE) *Every closed injective system has a fundamental limit space.*

Proof. Let $\{X_i, Y_i, f_i\}$ be an arbitrary CIS. Define $\widetilde{X} = X_0 \cup_{f_0} X_1 \cup_{f_1} X_2 \cup_{f_2} \cdots$ to be the quotient space obtained of the coproduct (or topological sum) $\coprod_{i=0}^{\infty} X_i$ by identifying each $Y_i \subset X_i$ with $f_i(Y_i) \subset X_{i+1}$. Define each $\widetilde{\varphi}_i : X_i \to \widetilde{X}$ to be the projection from X_i into quotient space $\widetilde{X}$. Then $\{\widetilde{X}, \widetilde{\varphi}_i\}$ is a fundamental limit space for the given CIS $\{X_i, Y_i, f_i\}$. $\square$

The latter two theorems implies that every CIS has, up to homeomorphisms, a unique fundamental limit space. This will be remembered and used many times in the article.

4 Examples of CIS's and limit spaces

In this section we will show some interesting examples of limit spaces. The first example is very simple and the second shows the existence of a limit space which is not a fundamental limit space. This example will be highlighted in the last section of this article by proving the essentiality of certain assumptions in the characterization of the fundamental limit space through the Hausdorff axiom. The other examples show known spaces as fundamental limit spaces.

Example 4.1. *Identity limit space.*

Let $\{X_i, Y_i, f_i\}$ be the CIS with $Y_i = X_i = X$ and $f_i = id_X$, for all $i \in \mathbb{N}$, where X is an arbitrary topological space and $id_X : X \to X$ is the identity map. It is easy to see that $\{X, id_X\}$ is a fundamental limite space for $\{X_i, Y_i, f_i\}$.

Example 4.2. *Existence of limit space which is not a fundamental limit space.*

Assume $X_0 = [0, 1)$ and $Y_0 = \{0\}$. Take $X_i = Y_i = [0, 1]$, for all $i \geq 1$. Let $f_0 : Y_0 \to X_1$ be the inclusion $f(0) = 0$ and $f_i = identity$, for all $i \geq 1$.

Consider the sphere S^1 as a subspace of $\mathbb{R}^2$. Define

$$\phi_0 : X_0 \to S^1, \text{ by } \phi_0(t) = (\cos \pi t, -\sin \pi t) \text{ and}$$

$$\phi_i : X_i \to S^1, \text{ by } \phi_i(t) = (\cos \pi t, \sin \pi t), \text{ for all } i \geq 1.$$

It is easy to see that $S^1 = \bigcup_{i=0}^{\infty} \phi_i(X_i)$ and each ϕ_i is an imbedding onto its image. Moreover, $\phi_i(X_i) \cap \phi_j(X_j) \doteq \phi_j f_{i,j-1}(Y_i)$, which implies the condition L.3.

Therefore, $\{S^1, \phi_i\}$ is a limit space for the CIS $\{X_i, Y_i, f_i\}$. However, this limit space is not a fundamental limit space, since $\phi_0(X_0)$ is not closed in S^1, (or again, since S^1 is compact though X_0 is not). (See Figure 1 below).

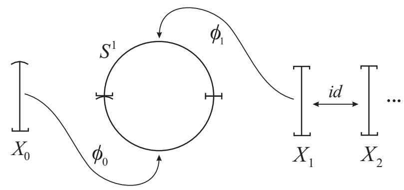

Figure 1: Limit space (not fundamental)

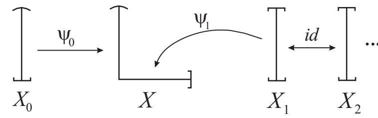

Figure 2: Fundamental limit space

Now, we consider the subspace $X = \{(x, 0) \in \mathbb{R}^2 : 0 \le x \le 1\} \cup \{(0, y) \in \mathbb{R}^2 : 0 \le y < 1\}$ of $\mathbb{R}^2$. Define

$$\psi_0 : X_0 \to X, \text{ by } \psi_0(t) = (0, t) \text{ and}$$

$$\psi_i : X_i \to X, \text{ by } \psi_i(t) = (t, 0), \text{ for all } i \ge 1.$$

We have $X = \bigcup_{i=0}^{\infty} \psi_i(X_i)$, where each ϕ_i is an imbedding onto its image, such that $\psi_i(X_i)$ is closed in X. Moreover, since $\psi_i(X_i) \cap \psi_j(X_j) \doteq \psi_j f_{i,j-1}(Y_i)$, it follows that $\{X, \psi_i\}$ is a fundamental limit space for the CIS $\{X_i, Y_i, f_i\}$. (See Figure 2 above).

(The bijection $\beta : S^1 \to X$ of the Theorem 2.10 is not continuous here).

Example 4.3. *The infinite-dimensional sphere S^∞.*

For each $n \in \mathbb{N}$, we consider the n-dimensional sphere

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\},$$

and the "equatorial inclusions" $f_n : S^n \to S^{n+1}$ given by

$$f_n(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n+1}, 0).$$

Then $\{S^n, S^n, f_n\}$ is a CIS. Its fundamental limit space is $\{S^\infty, \phi_n\}$, where S^∞ is the infinite-dimensional sphere and, for each $n \in \mathbb{N}$, the imbedding $\phi_n : S^n \to S^\infty$ is the natural "equatorial inclusion".

Example 4.4. *The infinite-dimensional torus T^∞.*

For each $n \ge 1$, we consider the n-dimensional torus $T^n = \prod_{i=1}^{n} S^1$ and the closed injections $f_n : T^n \to T^{n+1}$ given by $f_n(x_1, \dots, x_n) = (x_1, \dots, x_n, (1, 0))$, where each $x_i \in S^1$. Then $\{T^n, T^n, f_n\}$ is a CIS, whose fundamental limit space is $\{T^\infty, \phi_n\}$, where $T^\infty = \prod_{i=1}^{\infty} S^1$ is the infinite-dimensional torus and, for each $n \in \mathbb{N}$, the imbedding $\phi_n : T^n \to T^\infty$ is the natural inclusion

$$\phi_n(x_1, \dots, x_n) = (x_1, \dots, x_n, (1, 0), (1, 0), \dots).$$

Example 4.3 is a particular case the following one:

Example 4.5. *The CW-complexes as fundamental limit spaces for its skeletons.*

Let K be an arbitrary CW-complex. For each $n \in \mathbb{N}$, let K^n be the n-skeleton of K and consider the natural inclusions $l_n : K^n \to K^{n+1}$ of the n-skeleton into $(n+1)$-skeleton. If the dimension $\dim(K)$ *of K is finite, then we put $K^m = K$ and $l_m : K^m \to K^{m+1}$ to be the identity map, for all $m \ge \dim(K)$.*

It is known that a CW-complex has the weak topology with respect to their skeletons, that is, a subset $A \subset K$ is closed in K if and only if $A \cap K^n$ is closed in K^n for all n. Thus, $\{K^n, K^n, l_n\}$ is a CIS, whose fundamental limit space is $\{K, \phi_n\}$, where each $\phi_n : K^n \to K$ is the natural inclusions of the n-skeleton K^n into K.

For details of the CW-complex theory see [2] or [6].

The example below is a consequence of the previous one.

Example 4.6. *The infinite-dimensional projective space $\mathbb{R}\mathrm{P}^\infty$.*

There is always a natural inclusion $f_n : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n+1}$, which is a closed injective continuous map. ($\mathbb{R}\mathrm{P}^n$ is the n-skeleton of the $\mathbb{R}\mathrm{P}^{n+1}$). It follows that $\{\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n, f_n\}$ is a CIS. The fundamental limit space for this CIS is the infinite-dimensional projective space $\mathbb{R}\mathrm{P}^\infty$.

For details about infinite-dimensional sphere and projective plane see [2].

5 The category of closed injective systems and cis-morphisms

Let $\mathfrak{X} = \{X_i, Y_i, f_i\}_i$ and $\mathfrak{Z} = \{Z_i, W_i, g_i\}_i$ be two closed injective systems. By a **cis-morphism** $\mathfrak{h} : \mathfrak{X} \to \mathfrak{Z}$ we mean a collection

$$\mathfrak{h} = \{h_i : X_i \to Z_i\}_i$$

of closed continuous maps checking the following conditions:

1. $h_i(Y_i) \subset W_i$, for all $i \in \mathbb{N}$.

2. $h_{i+1} \circ f_i = g_i \circ h_i|_{Y_i}$, for all $i \in \mathbb{N}$.

This latter condition is equivalent to commutativity of the diagram below, for each $i \in \mathbb{N}$.

$$\begin{tikzcd} Y_i \arrow[r, "h_i|_{Y_i}"] \arrow[d, "f_i"'] & W_i \arrow[d, "g_i"] \\ X_{i+1} \arrow[r, "h_{i+1}"'] & Z_{i+1} \end{tikzcd}$$

We say that a cis-morphism $\mathfrak{h} : \mathfrak{X} \to \mathfrak{Z}$ is a **cis-isomorphism** if each map $h_i : X_i \to Z_i$ is a homeomorphism and carries Y_i homeomorphicaly onto W_i.

For each arbitrary CIS, say $\mathfrak{X} = \{X_i, Y_i, f_i\}_i$, there is an identity cis-morphism $\imath : \mathfrak{X} \to \mathfrak{X}$ given by $\imath_i : X_i \to X_i$ equal to identity map for each $i \in \mathbb{N}$.

Moreover, if $\mathfrak{h} : \mathfrak{X}^{(1)} \to \mathfrak{X}^{(2)}$ and $\mathfrak{k} : \mathfrak{X}^{(2)} \to \mathfrak{X}^{(3)}$ are two cis-morphisms, then it is clear that its natural composition

$$\mathfrak{k} \circ \mathfrak{h} : \mathfrak{X}^{(1)} \to \mathfrak{X}^{(3)}$$

is a cis-morphism from $\mathfrak{X}^{(1)}$ into $\mathfrak{X}^{(3)}$.

Also, it is easy to check that associativity of compositions holds whenever possible: if $\mathfrak{h} : \mathfrak{X}^{(1)} \to \mathfrak{X}^{(2)}$, $\mathfrak{k} : \mathfrak{X}^{(2)} \to \mathfrak{X}^{(3)}$ and $\mathfrak{r} : \mathfrak{X}^{(3)} \to \mathfrak{X}^{(4)}$, then

$$\mathfrak{r} \circ (\mathfrak{k} \circ \mathfrak{h}) = (\mathfrak{r} \circ \mathfrak{k}) \circ \mathfrak{h}.$$

This shows that the closed injective system and the cis-morphisms between they forms a category, which we denote by $\mathfrak{Cis}$. (See [3] for details on basic category theory).

Theorem 5.1. *Every inductive systems on the category* $\mathfrak{Cis}$ *admit limit.*

Proof. Let $\{\mathfrak{X}^{(n)}, \mathfrak{h}^{(mn)}\}_{m,n}$ be an inductive system of closed injective system and cis-morphisms. Then, each $\mathfrak{X}^{(n)}$ is of the form $\mathfrak{X}^{(n)} = \{X_i^{(n)}, Y_i^{(n)}, f_i^{(n)}\}_i$ and each $\mathfrak{h}^{(mn)} : \mathfrak{X}^{(m)} \to \mathfrak{X}^{(n)}$ is a cis-morphism and, moreover, $\mathfrak{h}^{(pq)} \circ \mathfrak{h}^{(qr)} = \mathfrak{h}^{(pr)}$, for all $p, q, r \in \mathbb{N}$.

For each $m \in \mathbb{N}$, we write $\mathfrak{h}^{(m)}$ to be $\mathfrak{h}^{(mn)}$ when $m = n + 1$.

For each $i \in \mathbb{N}$, we have the inductive system $\{X_i^{(n)}, h_i^{(mn)}\}_{m,n}$, that is, the injective system of the topological spaces $X_i^{(1)}, X_i^{(2)}, \ldots$ and all continuous maps $h_i^{(mn)} : X_i^{(m)} \to X_i^{(n)}$, $m, n \in \mathbb{N}$, of the collection $\mathfrak{h}^{(mn)}$.

Now, each inductive system $\{X_i^{(n)}, h_i^{(mn)}\}_{m,n}$ can be consider as the closed injective system $\{X_i^{(n)}, X_i^{(n)}, h_i^{(n)}\}_n$. Let $\{X_i, \xi_i^{(n)}\}_n$ be a fundamental limit space for $\{X_i^{(n)}, X_i^{(n)}, h_i^{(n)}\}_n$.

$$\begin{tikzcd}
 & & & X_i \\
{} \arrow[r, "h_i^{(qm)}"'] & X_i^{(m)} \arrow[urr, "\xi_i^{(m)}"] \arrow[r, "h_i^{(mn)}"'] & X_i^{(n)} \arrow[ur, "\xi_i^{(n)}"'] \arrow[r, "h_i^{(np)}"'] & {}
\end{tikzcd}$$

Then, each $\xi_i^{(n)} : X_i^{(n)} \to X_i$ is an imbedding, and we have $\xi_i^{(m)} = \phi_i^{(n)} \circ h_i^{(mn)}$ for all $m < n$. Moreover, X_i has a weak topologia induced by the collection $\{\xi_i^{(n)}\}_n$.

For any $m, n \in \mathbb{N}$, with $m \leq n$, we have

$$\xi_i^{(m)}(Y_i^{(m)}) = \xi_i^{(n)} \circ h_i^{(mn)}(Y_i^{(m)}) \subset \xi_i^{(n)}(Y_i^{(n)}),$$

by condition **1** of the definition of cis-morphism. Moreover, each $\xi_i^{(n)}(Y_i^{(n)})$ is closed in X_i, since each $\xi_i^{(n)}$ is an imbedding.

For each $i \in \mathbb{N}$, we define

$$Y_i = \bigcup_{n \in \mathbb{N}} \xi_i^{(n)}(Y_i^{(n)}).$$

Then, by preceding paragraph, Y_i is a union of linked closed sets, that is, Y_i is the union of the closed sets of the ascendent chain

$$\xi_i^{(1)}(Y_i^{(1)}) \subset \xi_i^{(2)}(Y_i^{(2)}) \subset \cdots \subset \xi_i^{(m)}(Y_i^{(m)}) \subset \xi_i^{(m+1)}(Y_i^{(m+1)}) \subset \cdots$$

Now, since $\{X_i, \xi_i^{(n)}\}_n$ is a fundamental limit space for $\{X_i^{(n)}, Y_i^{(n)}, h_i^{(n)}\}_n$, for each $m \in \mathbb{N}$, we have

$$(\xi_i^{(m)})^{-1}(Y_i) = (\xi_i^{(m)})^{-1}(\cup_{n \in \mathbb{N}} \xi_i^{(n)}(Y_i^{(n)})) = Y_i^m \text{ which is closed in } X_i^{(m)}.$$

Therefore, since X_i has the weak topology induced by the collection $\{\xi_i^{(n)}\}_n$, it follows that Y_i is closed in X_i.

Now, we will build, for each $i \in \mathbb{N}$, an injection $f_i : Y_i \to X_{i+1}$ making $\{X_i, Y_i, f_i\}_i$ a closed injective system. For each $i \in \mathbb{N}$, we have the diagram shown below.

For each $x \in \xi_i^{(n)}(Y_i^{(n)}) \subset X_i$, there is a unique $y \in Y_i^{(n)}$ such that $\xi_i^{(n)}(y) = x$. Then, we define $f_i(x) = (\xi_{i+1}^{(n)} \circ f_i^{(n)})(y)$.

$$\begin{tikzcd}
Y_i^{(n)} \arrow[r, "\xi_i^{(n)}"] \arrow[d, "f_i^{(n)}"'] & \xi_i^{(n)}(Y_i^{(n)}) \arrow[d, dotted, "f_i"] \\
X_{i+1}^{(n)} \arrow[r, "\xi_{i+1}^{(n)}"'] & X_{i+1}
\end{tikzcd}$$

It is clear that each $f_i : \xi_i^{(n)}(Y_i^{(n)}) \to X_{i+1}$ is a closed injective continuous map, since each ξ_i and $f_i^{(n)}$ are closed injective continuous maps.

Now, we define $f_i : Y_i \to X_{i+1}$ in the following way: For each $x \in Y_i$, there is an integer $n \in \mathbb{N}$ such that $x \in \xi_i^{(n)}(Y_i^n)$. Then, there is a unique $y \in Y_i^{(n)}$ such that $\xi_i^{(n)}(y) = x$. We define $f_i(x) = (\xi_{i+1}^{(n)} \circ f_i^{(n)})(y)$.

Each $f_i : Y_i \to X_{i+1}$ is well defined. In fact: suppose that x belong to $\xi_i^{(m)}(Y_i^m) \cap \xi_i^{(n)}(Y_i^n)$. Suppose, without loss of generality, that $m < n$. There are unique $y_m \in Y_i^m$ and $y_n \in Y_i^n$, such that $\xi_i^{(m)}(y_m) = y = \phi_i^{(n)}(y_n)$. Then, $y_n = h_i^{(mn)}(y_m)$. Thus,

$$\xi_{i+1}^{(n)} \circ f_i^{(n)}(y_n) = \xi_{i+1}^{(n)} \circ f_i^{(n)} \circ h_i^{(mn)}(y_m) = \xi_{i+1}^{(n)} \circ h_{i+1}^{(mn)} \circ f_i^{(m)}(y_m) = \xi_{i+1}^{(m)} \circ f_i^{(m)}(y_m).$$

Now, since each $f_i : Y_i \to X_{i+1}$ is obtained of a collection of closed injective continuous maps which coincides on closed sets, it follows that each f_i is a closed injective continuous map.

This proves that $\{X_i, Y_i, f_i\}_i$ is a closed injective system. Denote it by $\mathfrak{X}$.

For each $n \in \mathbb{N}$, let $\mathcal{E}^{(n)} : \mathfrak{X}^{(n)} \to \mathfrak{X}$ be the collection

$$\mathcal{E}^{(n)} = \{\xi_i^{(n)} : X_i^{(n)} \to X_i\}_i.$$

It is clear by the construction that $\mathcal{E}^{(n)}$ is a cis-morphism from $\mathfrak{X}^{(n)}$ into $\mathfrak{X}$. Moreover, we have $\mathcal{E}^{(m)} = \langle^{(mn)} \circ \mathcal{E}^{(n)}$. Therefore, $\{\mathfrak{X}, \mathcal{E}^{(n)}\}_n$ is a direct limit for the inductive system $\{\mathfrak{X}^{(n)}, \mathfrak{h}^{(mn)}\}_{m,n}$. ☐

6 The transition to fundamental limit space as a functor

Henceforth, we will write $\mathfrak{Top}$ to denote the category of the topological spaces and continuous maps.

For each CIS $\mathfrak{X} = \{X_i, Y_i, f_i\}_i$, we will denote its fundamental limite space by $\mathcal{L}(\mathfrak{X})$. The passage to the fundamental limit defines a function

$$\mathcal{L} : \mathfrak{Cis} \longrightarrow \mathfrak{Top}$$

which associates to each CIS $\mathfrak{X}$ its fundamental limit space $\mathcal{L}(\mathfrak{X}) = \{X, \phi_i\}$.

Theorem 6.1. *Let $\mathfrak{h} : \mathfrak{X} \to \mathfrak{Z}$ be a cis-morphism between closed injective systems, and let $\mathcal{L}(\mathfrak{X}) = \{X, \phi_i\}_i$ and $\mathcal{L}(\mathfrak{Z}) = \{Z, \psi_i\}_i$ be the fundamental limit spaces for $\mathfrak{X}$ and $\mathfrak{Z}$, respectively. Then, there is a unique closed continuous map $\mathcal{L}\mathfrak{h} : X \to Z$ such that $\mathcal{L}\mathfrak{h} \circ \phi_i = \psi_i \circ h_i$, for all $i \in \mathbb{N}$.*

Proof. Write $\mathfrak{h} = \{h_i : X_i \to Z_i\}_i$. We define the map $\mathcal{L}\mathfrak{h} : X \to Z$ as follows: First, consider $\mathcal{L}(\mathfrak{X}) = \{X, \phi_i\}$ and $\mathcal{L}(\mathfrak{Z}) = \{Z, \psi_i\}$. For each $x \in X$, there is $x_i \in X_i$, for some $i \in \mathbb{N}$, such that $x = \phi_i(x_i)$. Then, we define

$$\mathcal{L}\mathfrak{h}(x) = \psi_i \circ h_i(x_i).$$

This map is well defined. In fact, if $x = \phi_i(x_i) = \phi_j(x_j)$, with $i < j$, then $x \in \phi_i(X_i) \cap \phi_j(X_j) \doteq \phi_j f_{i,j-1}(Y_{i,j-1})$ and $x_j = f_{i,j-1}(x_i)$. Thus,

$$\psi_j \circ h_j(x_j) = \psi_j \circ h_j \circ f_{i,j-1}(x_i) = \psi_j \circ g_{i,j-1} \circ h_i(x_i) = \psi_i \circ h_i(x_i).$$

Now, since $\mathcal{L}\mathfrak{h}$ is obtained from a collection of closed continuous maps which coincide on closed sets, it is easy to see that $\mathcal{L}\mathfrak{h}$ is a closed continuous map.

Moreover, it is easy to see that $\mathcal{L}\mathfrak{h}$ is the unique continuous map from X into Z which verifies, for each $i \in \mathbb{N}$, the commutativity $\mathcal{L}\mathfrak{h} \circ \phi_i = \psi_i \circ h_i$. ☐

Sometimes, we write $\pounds\mathfrak{h} : \pounds(\mathfrak{X}) \to \pounds(\mathfrak{Z})$ instead $\pounds\mathfrak{h} : X \to Y$. This map is called the **fundamental map** induced by $\mathfrak{h}$.

Corollary 6.2. *The transition to the fundamental limit space is a functor from the category* $\mathfrak{Cis}$ *into the category* $\mathfrak{Top}$.

For details on functors see [3].

Corollary 6.3. *If* $\mathfrak{h} : \mathfrak{X} \to \mathfrak{Z}$ *is a cis-isomorphism, then the fundamental map* $\pounds\mathfrak{h} : \pounds(\mathfrak{X}) \to \pounds(\mathfrak{Z})$ *is a homeomorphism.*

This implies that isomorphic closed injective systems have homeomorphic fundamental limit spaces.

7 Compatibility of limits

In this section, given a CIS $\mathfrak{X} = \{X_i, Y_i, f_i\}$ with fundamental limit space $\{X, \phi_i\}$, sometimes we write $\pounds(\mathfrak{X})$ to denote only the topological space X. This is clear in the context.

Theorem 7.1. *Let* $\{\mathfrak{X}^{(n)}, \mathfrak{h}^{(mn)}\}_{m,n}$ *be an inductive system on the category* $\mathfrak{Cis}$ *and let* $\{\mathfrak{X}, \mathcal{E}^{(n)}\}_n$ *its direct limit. Then* $\{\pounds(\mathfrak{X}^{(n)}), \pounds\mathfrak{h}^{(mn)}\}_{m,n}$ *is an inductive system on the category* $\mathfrak{Top}$*, which admits* $\pounds(\mathfrak{X})$ *as its directed limit homeomorphic.*

Proof. By uniqueness of the direct limit, we can assume that $\{\mathfrak{X}, \Phi^{(n)}\}_n$ is the direct limit constructed in the proof of the Theorem 5.1. Then, we have

$$\mathcal{E}^{(n)} : \mathfrak{X}^{(n)} \to \mathfrak{X} \text{ given by } \mathcal{E}^{(n)} = \{\xi_i^{(n)} : X_i^{(n)} \to X_i\}_i,$$

where $\{X_i, \xi_i^{(n)}\}_n$ is a fundamental limit space for $\{X_i^{(n)}, X_i^{(n)}, h_i^{(n)}\}_n$.

By the Theorem 6.1, $\{\pounds(\mathfrak{X}^{(n)}), \pounds\mathfrak{h}^{(mn)}\}_{m,n}$ is a inductive system.

For each $n \in \mathbb{N}$, write $\mathfrak{X}^{(n)} = \{X_i^{(n)}, Y_i^{(n)}, f_i^{(n)}\}_i$ and $\pounds(\mathfrak{X}^n) = \{X^{(n)}, \phi_i^{(n)}\}_i$. Moreover, write $\mathfrak{X} = \{X_i, Y_i, f_i\}_i$ and $\pounds(\mathfrak{X}) = \{X, \phi_i\}_i$. Then, the inductive system $\{\pounds(\mathfrak{X}^{(n)}), \pounds\mathfrak{h}^{(mn)}\}_{m,n}$ can be write as $\{X^{(n)}, \pounds\mathfrak{h}^{(mn)}\}_{m,n}$.

We need to show that there is a collection of maps $\{\vartheta^{(n)} : X^{(n)} \to X\}_n$ such that $\{X, \vartheta^{(n)}\}_n$ is a direct limit for the system $\{X^{(n)}, \pounds\mathfrak{h}^{(mn)}\}_{m,n}$.

For each $x \in X^{(n)}$, there is a point $x_i \in X_i^{(n)}$, for some $i \in \mathbb{N}$, such that $x = \phi_i^{(n)}(x_i)$. We define $\vartheta^{(n)} : X^{(n)} \to X$ by $\vartheta^{(n)}(x) = \phi_i \circ \xi_i^{(n)}(x_i)$.

The map $\vartheta^{(n)}$ is well defined. In fact: If $x = \phi_i^{(n)}(x_i) = \phi_j^{(n)}(x_j)$, with $i \leq j$, then $x \in \phi_i^{(n)}(X_i^{(n)}) \cap \phi_j^{(n)}(X_j^{(n)}) \doteq \phi_j^{(n)} f_{i,j-1}^{(n)}(Y_{i,j-1}^{(n)})$ and $x_j = f_{i,j-1}^{(n)}(x_i)$ and $x_i \in Y_{i,j} \subset X_i$. Now, in the diagram below, the two triangles and the big square are commutative. In it, we write $\xi_i^{(n)}|$ and $\phi_i^{(n)}|$ to denote the obvious restriction.

It follows that

$$\phi_j \circ \xi_j^{(n)}(x_j) = \phi_j \circ \xi_j^{(n)} \circ f_{i,j-1}(n)(x_i) = \phi_j \circ f_{i,j-1} \circ \xi_i^{(n)}(x_i) = \phi_i \circ \xi_i^{(n)}(x_i).$$

It is sufficient to prove that the map $\vartheta^{(n)}$ is well defined. Moreover, note that this map makes the diagram above in a commutative diagram.

$$\begin{tikzcd}
X_i \arrow[rr, "f_{i,j-1}"] \arrow[dr, "\phi_i"'] & & X_j \arrow[dl, "\phi_j"] \\
& X & \\
& X^{(n)} \arrow[u, dotted, "\vartheta^{(n)}"] & \\
Y^{(n)}_{i,j} \arrow[uuu, "\xi_i^{(n)}|"] \arrow[ur, "\phi_i^{(n)}|"] \arrow[rr, "f^{(n)}_{i,j-1}"'] & & X_j^{(n)} \arrow[uuu, "\xi_j^{(n)}"'] \arrow[ul, "\phi_j^{(n)}"']
\end{tikzcd}$$

Now, by the Theorem 6.1 we have $\pounds\mathfrak{h}^{(mn)} \circ \phi_i^{(m)} = \phi_i^{(n)} \circ h_i^{(n)}$ for all integers $m < n$, since $\pounds(\mathfrak{X}^n) = \{X^{(n)}, \phi_i^{(n)}\}_i$.

Let $x \in X^{(m)}$ be an arbitrary point. Then, there is $x_i \in X_i^{(m)}$ such that $x = \phi_i^{(m)}(x_i)$. Also, for all $n \in \mathbb{N}$ with $m < n$, we have $\pounds\mathfrak{h}^{(mn)}(x) = \phi_i^{(n)} \circ h_i^{(mn)}(x_i)$. Thus, we have,

$$\vartheta^{(n)} \circ \pounds\mathfrak{h}^{(mn)}(x) = \phi_i \circ \xi_i^{(n)}(h_i^{(mn)}(x_i)) = \phi_i \circ \xi^{(m)}(x_i) = \vartheta^{(m)}(x).$$

This shows that $\vartheta^{(n)} \circ \pounds\mathfrak{h}^{(mn)} = \vartheta^{(m)}$ for all integers $m < n$.

Let A be a closed subset of X. Then it is clear that $(\phi_i \circ \xi_i^{(n)})^{-1}(A)$ is closed in $X_i^{(n)}$, since ϕ_i and $\xi_i^{(n)}$ are continuous maps. Now, we have $(\vartheta^{(n)})^{-1}(A) = \phi_i^{(n)}((\phi_i \circ \xi_i^{(n)})^{-1}(A))$. Then, since $\phi_i^{(n)}$ is an imbedding (and so a closed map), it follows that $(\vartheta^{(n)})^{-1}(A)$ is a closed subset of $X^{(n)}$. Therefore, $\vartheta^{(n)}$ is a continuous.

Now, it is not difficult to prove that $\{X, \vartheta^{(n)}\}_n$ satisfies the universal mapping problem (see [3]). This concludes the proof. $\square$

8 Inductive closed injective systems

In this section, we will study a particular kind of closed injective systems, which has some interesting properties. More specifically, we study the CIS's of the form $\{X_i, X_i, f_i\}$, which are called **inductive closed injective system**, or an inductive CIS, to shorten.

In an inductive CIS $\{X_i, X_i, f_i\}$, any two injections f_i and f_j, with $i < j$, are **componible**, that is, the composition $f_{i,j} = f_j \circ \cdots \circ f_i$ is always defined throughout domine X_i of f_i.

Hence, fixing $i \in \mathbb{N}$, for each $j > i$ we have a closed injection $f_{i,j} : X_i \to X_{j+1}$. Because this, we define, for each $i < j \in \mathbb{N}$,

$$f_i^i = id_{X_i} : X_i \to X_i$$

$$f_i^j = f_{i,j-1} : X_i \to X_j.$$

By this definition, it follows that $f_i^k = f_j^k \circ f_i^j$, for all $i \leq j \leq k$. Therefore, $\{X_i, f_i^j\}$ is an **inductive system** on the category $\mathfrak{Top}$.

We will construct a direct limit for this inductive system.

Let $\coprod X_i = \coprod_{i=0}^{\infty} X_i$ be the coproduct (or topological sum) of the spaces X_i.

Consider the canonical inclusions $\varphi_i : X_i \to \coprod X_i$. It is obvious that each φ_i is a homeomorphism onto its image.

Over the space $\coprod X_i$, consider the relation $\sim$ defined by:

$$x \sim y \Leftrightarrow \begin{cases} \exists\ x_i \in X_i, y_j \in X_j \text{ with } x = \varphi_i(x_i) \text{ e } y = \varphi_j(y_j), \text{ such that} \\ y_j = f_i^j(x_i) \text{ if } i \leq j \text{ and } x_i = f_j^i(y_j) \text{ if } j < i. \end{cases}.$$

Lemma 8.1. *The relation $\sim$ is an equivalence relation over $\coprod X_i$.*

Proof. We need to check the veracity of the properties reflexive, symmetric and transitive.

Reflexive: Let $x \in X$ be a point. There is $x_i \in X_i$ such that $x = \psi_i(x_i)$, for some $i \in \mathbb{N}$. We have $x_i = f_i^i(x_i)$. Therefore $x \sim x$.

Symmetric: It is obvious by definition of the relation $\sim$.

Transitive: Assume that $x \sim y$ and $y \sim z$. Suppose that $x = \varphi_i(x_i)$ and $y = \varphi_j(y_j)$ with $y_j = f_i^j(x_i)$. In this case, $i \leq j$. (The other case is analogous and is omitted). Since $y \sim z$, we can have:

Case 1 : $y = \varphi_j(y'_j)$ and $z = \varphi_k(z_k)$ with $j \leq k$ and $z_k = f_j^k(y'_j)$. Then $\varphi_j(y_j) = y = \varphi_j(y'_j)$, and so $y_j = y'_j$. Since $i \leq j \leq k$, we have $z_k = f_j^k(y_j) = f_j^k f_i^j(x_i) = f_i^k(x_i)$. Therefore $x \sim z$.

Case 2: $y = \varphi_j(y'_j)$ and $z = \varphi_k(z_k)$ with $k < j$ and $y'_j = f_k^j(z_k)$. Then $y_j = y'_j$, as before. Now, we have again two possibility:

(a) If $i \leq k < j$, then we have $f_k^j(z_k) = y_j = f_i^j(x_i) = f_k^j f_i^k(x_i)$. Thus $z_k = f_i^k(x_i)$ and $x \sim z$.

(b) If $k < i \leq j$, then we have $f_i^j(x_i) = y_j = f_k^j(z_k) = f_i^j f_k^i(z_k)$. Thus $x_i = f_k^i(z_k)$ and $x \sim z$. $\square$

Let $\widetilde{X} = (\coprod X_i)/\sim$ be the quotient space obtained of $\coprod X_i$ by the equivalence relation $\sim$, and for each $i \in \mathbb{N}$, let $\widetilde{\varphi}_i : X_i \to \widetilde{X}$ be the composition $\widetilde{\varphi}_i = \rho \circ \varphi_i$, where $\rho : \coprod X_i \to \widetilde{X}$ is the quotient projection.

$$\widetilde{\varphi}_i : X_i \xrightarrow{\varphi_i} \coprod X_i \xrightarrow{\rho} \widetilde{X}$$

Note that, since $\widetilde{X}$ has the quotient topology induced by projection ρ, a subset $A \subset \widetilde{X}$ is closed in $\widetilde{X}$ if and only if $\widetilde{\varphi}_i^{-1}(A)$ is close in X_i, for each $i \in \mathbb{N}$.

Given $x, y \in \coprod X_i$ with $x, y \in X_i$, then $x \sim y \Leftrightarrow x = y$. Thus, each $\widetilde{\varphi}_i$ is one-to-one fashion onto $\widetilde{\varphi}_i(X_i)$. Moreover, it is obvious that $\widetilde{X} = \cup_{i=0}^{\infty} \widetilde{\varphi}_i(X_i)$.

These observations suffice to conclude the following:

Theorem 8.2. *$\{\widetilde{X}, \widetilde{\varphi}_i\}$ is a fundamental limit space for the inductive CIS $\{X_i, X_i, f_i\}$. Moreover, $\{\widetilde{X}, \widetilde{\varphi}_i\}$ is a direct limit for the inductive system $\{X_i, f_i^j\}$.*

For details on direct limit see [3].

Remark 8.3. *If we consider an arbitrary CIS $\{X_i, Y_i, f_i\}$, then the relation $\sim$ is again an equivalence relation over the coproduct $\coprod X_i$. Moreover, in this circumstances, if $\varphi_i(x_i) = x \sim y = \varphi_j(y_j)$, then we must have:*

(a) *If $i = j$, then $x = y$.*

(b) *If $i < j$, then f_i and f_{j-1} are semicomponible and $x_i \in Y_{i,j-1}$;*

(c) *If $i > j$, then f_j and f_{i-1} are semicomponible and $y_j \in Y_{j,i-1}$.*

Therefore, it follows that the space $\widetilde{X} = (\coprod X_i)/\sim$ is exactly the attaching space $X_0 \cup_{f_0} X_1 \cup_{f_1} X_2 \cup_{f_2} \cdots$, and the maps $\widetilde{\varphi}_i$ are the projections of X_i into this space, as in theorem of the existence of fundamental limit space (Theorem 3.6).

9 Functoriality on fundamental limit spaces

Let $\mathbf{F} : \mathfrak{Top} \to \mathfrak{Mod}$ be a functor of the category $\mathfrak{Top}$ into the category $\mathfrak{Mod}$ (the category of the R-modules and R-homomorphisms), where R is a commutative ring with identity element. (See [3]).

Let $\{X_i, X_i, f_i\}$ be an arbitrary inductive CIS, and consider the inductive system $\{X_i, f_i^j\}$ constructed in the previous section. The functor $\mathbf{F}$ turns this system into the inductive system $\{\mathbf{F}X_i, \mathbf{F}f_i^j\}$ on the category $\mathfrak{Mod}$.

Theorem 9.1. (Of the Functorial Invariance) *Let $\{X, \phi_i\}$ be a fundamental limit space for the inductive CIS $\{X_i, X_i, f_i\}$ and let $\{M, \psi_i\}$ be a direct limit for $\{\mathbf{F}X_i, \mathbf{F}f_i^j\}$. Then, there is a unique R-isomorphism $h : \mathbf{F}X \to M$ such that $\psi_i = h \circ \mathbf{F}\phi_i$, for all $i \in \mathbb{N}$.*

Proof. By the Theorem 8.2 and by uniqueness of fundamental limit space, there is a unique homeomorphism $\beta : X \to \widetilde{X}$ such that $\widetilde{\varphi}_i = \beta \circ \phi_i$, for all $i \in \mathbb{N}$. Hence, $\mathbf{F}\beta : \mathbf{F}X \to \mathbf{F}\widetilde{X}$ is the unique R-isomorphism such that $\mathbf{F}\widetilde{\varphi}_i = \mathbf{F}\beta \circ \mathbf{F}\phi_i$.

Since $\{\widetilde{X}, \widetilde{\varphi}_i\}$ is a direct limit for the inductive system $\{X_i, f_i^j\}$ on the category $\mathfrak{Top}$, it follows that $\{\mathbf{F}\widetilde{X}, \mathbf{F}\varphi_i\}$ is a direct limit of the system $\{\mathbf{F}X_i, \mathbf{F}f_i^j\}$ on the category $\mathfrak{Mod}$. By universal property of direct limit, there is a unique R-isomorphism $\omega : \mathbf{F}\widetilde{X} \to M$ such that $\psi_i = \omega \circ \mathbf{F}\widetilde{\varphi}_i$.

Then, we take $h : \mathbf{F}X \to M$ to be the composition $h = \omega \circ \mathbf{F}\beta$. □

The universal property of direct limits among others properties can be found, for example, in Chapter 2 of [3].

Now, we describe some basic applications of the Theorem of the Functorial Invariance.

Example 9.2. *Let K be an arbitrary CW-complex and let $\{K^n, K^n, l_n\}$ be the CIS as in Example 4.5. It is clear that this CIS is an inductive CIS. Let $\mathbf{F} : \mathfrak{Top} \to \mathfrak{Mod}$ be an arbitrary functor. Given $m < n$ in $\mathbb{N}$, write l_m^n to denote the composition $l_{n-1} \circ \cdots \circ l_m : K^m \to K^n$. Then, $\{\mathbf{F}K^n, \mathbf{F}l_m^n\}$ is an inductive system on the category $\mathfrak{Mod}$. By Theorem 9.1, its direct limit is isomorphic to $\mathbf{F}K$.*

Example 9.3. *Homology of the sphere S^∞.*

Let $\{S^n, S^n, f_n\}$ be the CIS of Example 4.3. Its fundamental limit space is the infinite-dimensional sphere S^∞. Let $p > 0$ be an arbitrary integer. By previous example, $H_p(S^\infty)$ is isomorphic to direct limit of inductive system $\{H_p(S^n), H_p(f_m^n)\}$, where $f_m^n = f_{n-1} \circ \cdots \circ f_m : S^m \to S^n$, for $m \leq n$. Now, since $H_p(S^n) = 0$ for $n > p$, it follows that $H_p(S^\infty) = 0$, for all $p > 0$.

Details on homology theory can be found in [1], [2] and [5].

Example 9.4. *The infinite projective space $\mathbb{R}\mathrm{P}^\infty$ is a $K(\mathbb{Z}_2, 1)$ space.*

We know that $\pi_1(\mathbb{R}\mathrm{P}^n) \approx \mathbb{Z}_2$ for all $n \geq 2$ and $\pi_1(\mathbb{R}\mathrm{P}^1) \approx \mathbb{Z}$. Moreover, for integers $m < n$, the natural inclusion $f_m^n : \mathbb{R}\mathrm{P}^m \hookrightarrow \mathbb{R}\mathrm{P}^n$ induces a isomorphism $(f_m^n)_\# : \pi_1(\mathbb{R}\mathrm{P}^m) \approx \pi_1(\mathbb{R}\mathrm{P}^n)$. For details see [2].

The fundamental limit space for the CIS $\{\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n, f_n\}$ of the Example 4.6 is the infinite projective space $\mathbb{R}\mathrm{P}^\infty$. By Example 9.2, we have that $\pi_1(\mathbb{R}\mathrm{P}^\infty)$ is isomorphic to direct limit for the inductive system $\{\pi_1(\mathbb{R}\mathrm{P}^n), (f_m^n)_\#\}$. Then, by previous arguments it is easy to check that $\pi_1(\mathbb{R}\mathrm{P}^\infty) \approx \mathbb{Z}_2$.

On the other hand, for all $r > 1$, we have $\pi_r(\mathbb{R}\mathrm{P}^n) \approx \pi_r(S^n)$ for all $n \in \mathbb{N}$ (see [2]). Then, $\pi_r(S^n) = 0$ always that $1 < r < n$. Thus, it is easy to check that $\pi_r(\mathbb{R}\mathrm{P}^\infty) = 0$, for all $r > 1$.

For details on homotopy theory and $K(\pi, 1)$ spaces see [2] and [6].

Example 9.5. *The homotopy groups of S^∞.*

Since $\pi_r(S^n) = 0$ for all integers $r < n$, it is very easy to prove that $\pi_r(S^\infty) = 0$, for all $r \geq 1$.

Example 9.6. *The homology of the torus T^∞.*

Some arguments very simple and similar to above can be used to prove that $H_0(T^\infty) \approx R$ and $H_p(T^\infty) \approx \bigoplus_{i=1}^\infty R$, for all integer $p > 0$.

10 Counter-Funtoriality on fundamental limit spaces

Let $\mathbf{G} : \mathfrak{Top} \to \mathfrak{Mod}$ be a counter-functor from the category $\mathfrak{Top}$ into the category $\mathfrak{Mod}$, where R is a commutative ring with identity element.

Let $\{X_i, X_i, f_i\}$ be an arbitrary inductive CIS and consider the inductive system $\{X_i, f_i^j\}$ as before. The counter-functor $\mathbf{G}$ turns this inductive system on the category $\mathfrak{Top}$ into the reverse system $\{\mathbf{G}X_i, \mathbf{G}f_i^j\}$ on the category $\mathfrak{Mod}$.

Theorem 10.1. (Of the Counter-Functorial Invariance) *Let $\{X, \phi_i\}$ be a fundamental limit space for the inductive CIS $\{X_i, X_i, f_i\}$ and let $\{M, \psi_i\}$ be an inverse limit for $\{\mathbf{G}X_i, \mathbf{G}f_i^j\}$. Then there is a unique R-isomorphism $h : M \to \mathbf{G}X$ such that $\psi_i = \mathbf{G}\phi_i \circ h$, for all $i \in \mathbb{N}$.*

Proof. By the Theorem 8.2 and by uniqueness of fundamental limit space, there is a unique homeomorohism $\beta : X \to \widetilde{X}$ such that $\widetilde{\varphi}_i = \beta \circ \phi_i$, for all $i \in \mathbb{N}$. Hence, $\mathbf{G}\beta : \mathbf{G}\widetilde{X} \to \mathbf{G}X$ is the unique R-isomorphism such that $\mathbf{G}\widetilde{\varphi}_i = \mathbf{G}\phi_i \circ \mathbf{G}\beta$.

Since $\{\widetilde{X}, \widetilde{\varphi}_i\}$ is a direct limit for the inductive system $\{X_i, f_i^j\}$ on the category $\mathfrak{Top}$, it follows that $\{\mathbf{G}\widetilde{X}, \mathbf{G}\varphi_i\}$ is an inverse limit for the inverse system $\{\mathbf{G}X_i, \mathbf{G}f_i^j\}$ on the category $\mathfrak{Mod}$. By universal property of inverse limit, there is a unique R-isomorphism $\omega : M \to \mathbf{G}\widetilde{X}$ such that $\psi_i = \mathbf{G}\widetilde{\varphi}_i \circ \omega$.

Then, we take $h : M \to \mathbf{G}X$ to be the compost R-isomorphism $h = \mathbf{G}\beta \circ \omega$. ∎

The property of the inverse limit can be found in [3].

Now, we describe some basic applications of the Theorem of the Counter-Functorial Invariance.

Example 10.2. *Cohomology of the sphere S^∞.*

Since $H^p(S^n; R) \approx H_p(S^n; R)$ for all $p, n \in \mathbb{Z}$, it follows by the Theorem 10.1 and Example 9.3 that $H^0(S^\infty; R) \approx R$ and $H^p(S^\infty; R) = 0$, for all $p > 0$.

Example 10.3. *The cohomology of the torus T^∞.*

Since the homology and cohomology modules of a finite product of spheres are isomorphic, it follows by Theorem the 10.1 and Example 9.6 that $H^0(T^\infty) \approx R$ and $H^p(T^\infty) \approx \bigoplus_{i=1}^{\infty} R$, for all $p > 0$.

11 Finitely semicomponible and stationary CIS's

We say that a CIS $\{X_i, Y_i, f_i\}$ is **finitely semicomponible** if, for all $i \in \mathbb{N}$, there is only a finite number of indices $j \in \mathbb{N}$ such that f_i and f_j (or f_j and f_i) are semicomponible, that is, there is not an infinity sequence $\{f_k\}_{k \geq i_0}$ of semicomponible maps. Obviously, $\{X_i, Y_i, f_i\}$ is finitely semicomponible if and only if for some (so for all) limit space $\{X, \phi_i\}$ for $\{X_i, Y_i, f_i\}$, the collection $\{\phi_i(X_i)\}_i$ is a pointwise finite cover of X (that is, each point of X belongs to only a finite number of $\phi_i(X_i)'s$).

We say that a CIS $\{X_i, Y_i, f_i\}$ is **stationary** if there is a nonegative integer n_0 such that, for all $n \geq n_0$, we have $Y_n = Y_{n_0} = X_{n_0} = X_n$ and $f_n = \textit{identity map}$.

This section of text is devoted to the study and characterization of the limit space of these two special types of CIS's.

Theorem 11.1. *Let $\{X, \phi_i\}$ be an arbitrary limit space for the CIS $\{X_i, Y_i, f_i\}$. If the collection $\{\phi_i(X_i)\}_i$ is a locally finite cover of X, then $\{X_i, Y_i, f_i\}$ is finitely semicomponible. The reciprocal is true if $\{X, \phi_i\}$ is a fundamental limit space.*

Proof. The first part is trivial, since if the collection $\{\phi_i(X_i)\}_i$ is a locally finite cover of X, then it is a pointwise finite cover of X.

Suppose that $\{X, \phi_i\}$ is a fundamental limit space for the finitely semicomponible CIS $\{X_i, Y_i, f_i\}$. Let $x \in X$ be an arbitrary point. Then, there are nonnegative integers $n_0 \leq n_1$ such that $\phi_i^{-1}(\{x\}) \neq \emptyset \Leftrightarrow n_0 \leq i \leq n_1$. For each $n_0 \leq i \leq n_1$, write x_i to be the single point of X_i such that $x = \phi_i(x_i)$. It follows that $x_i \in Y_{n_i}$ for $n_0 \leq i \leq n_1 - 1$, but $x_{n_1} \notin Y_{n_1}$ and $x_{n_0} \notin f_{n_0-1}(Y_{n_0-1})$.

Since $f_{n_0-1}(Y_{n_0-1})$ is closed in X_{n_0} and $x_{n_0} \notin f_{n_0-1}(Y_{n_0-1})$, we can choose an open neighborhood V_{n_0} of x_{n_0} in X_{n_0} such that $V_{n_0} \cap f_{n_0-1}(Y_{n_0-1}) = \emptyset$.

Similarly, since $x_{n_1} \notin Y_{n_1}$ and Y_{n_1} is closed in X_{n_1}, we can choose an open neighborhood V_{n_1} of x_{n_1} in X_{n_1} such that $V_{n_1} \cap Y_{n+1} = \emptyset$.

Define $V = \phi_{n_0}(V_{n_0}) \cup \phi_{n_0+1}(X_{n_0+1}) \cup \cdots \cup \phi_{n_1-1}(X_{n_1-1}) \cup \phi_{n_1}(V_{n_1})$.

It is clear that $x \in V \subset X$ and $V \cap \phi_j(X_j) = \emptyset$ for all $j \notin \{n_0, \ldots, n_1\}$. Moreover, we have

$$\phi_j^{-1}(X - V) = \begin{cases} X_{n_0} - V_{n_0} & \text{if} \quad j = n_0 \\ X_{n_1} - V_{n_1} & \text{if} \quad j = n_1 \\ \emptyset & \text{if} \quad n_0 < j < n_1 \\ X_j & \text{otherwise} \end{cases}.$$

In all cases, we see that $\phi_j^{-1}(X - V)$ is closed in X_j. Thus, $X - V$ is closed in X. Therefore, we obtain an open neighborhood V of x which intersects only a finite number of $\phi_i(X_i)'s$. □

The reciprocal of the previous proposition is not true, in general, when $\{X, \phi_i\}$ is not a fundamental limit space. In fact, we have the following example in which the above reciprocal failure.

Example 11.2. *Consider the topological subspaces $X_0 = [1, 2]$ and $X_n = [\frac{1}{n+1}, \frac{1}{n}]$, for $n \geq 1$, of the real line $\mathbb{R}$, and take $Y_0 = \{1\}$ and $Y_n = \{\frac{1}{n+1}\}$ for $n \geq 1$. Define $f_n : Y_n \to X_{n+1}$ to be the natural inclusion, for all $n \in \mathbb{N}$. It is clear that the CIS $\{X_n, Y_n, f_n\}$ is finitely semicomponible, and its fundamental limit space is, up to homeomorphism, the subspace $X = (0, 2]$ of the real line, together the collection of natural inclusions $\phi_n : X_n \to X$. It is also obvious that the collection $\{\phi_i(X_i)\}_i$ is a locally finite cover of X. On the other hand, take*

$$Z = ((0, 1] \times \{0\}) \cup \{(1 + cos(\pi t - \pi), \sin(\pi t - \pi)) \in \mathbb{R}^2 : t \in [1, 2]\}.$$

Consider Z as a subspace of the $\mathbb{R}^2$. Then Z is homeomorphic to the sphere S^1. Consider the maps $\psi_0 : X_0 \to Z$ given by $\psi_0(t) = (1 + cos(\pi t - \pi), \sin(\pi t - \pi))$, and $\psi_n : X_n \to Z$ given by $\psi_n(t) = (t, 0)$, for all $n \geq 1$. It is easy to see that $\{Z, \psi_n\}$ is a limit space for the CIS $\{X_n, Y_n, f_n\}$. Now, note that the point $(0, 0) \in Z$ has no open neighborhood intercepting only a finite number of $\psi_n(X_n)'s$.

Theorem 11.3. *Let $\{X, \phi_i\}$ be a limit space for the CIS $\{X_i, Y_i, f_i\}$ and suppose that the collection $\{\phi_i(X_i)\}_i$ is a locally finite closed cover of X. Then $\{X, \phi_i\}$ is a fundamental limit space.*

Proof. We need to prove that a subset A of X is closed in X if and only if $\phi_i^{-1}(A)$ is closed in X_i for all $i \in N$.

If $A \subset X$ is closed in X, then it is clear that $\phi_i^{-1}(A)$ is closed in X_i for each $i \in \mathbb{N}$, since each ϕ_i is a continuous map.

Now, let A be a subset of X such that $\phi_i^{-1}(A)$ is closed in X_i, for all $i \in \mathbb{N}$. Then, since each ϕ_i is a imbedding, it follows that $\phi_i(\phi_i^{-1}(A)) = A \cap \phi_i(X_i)$ is closed in $\phi_i(X_i)$. But by hypothesis, $\phi_i(X_i)$ is closed in X. Therefore $A \cap \phi_i(X_i)$ is closed in X, for each $i \in \mathbb{N}$.

Let $x \in X - A$ be an arbitrary point and choose an open neighborhood V of x in X such that $V \cap \phi_i(X_i) \neq \emptyset \Leftrightarrow i \in \Lambda$, where $\Lambda \subset \mathbb{N}$ is a finite subset of indices. It follows that

$$V \cap A = \bigcup_{i \in \Lambda} V \cap A \cap \phi_i(X_i).$$

Now, since each $A \cap \phi_i(X_i)$ is closed in X and $x \notin A \cap \phi_i(X_i)$, we can choose, for each $i \in \Lambda$, an open neighborhood $V_i \subset V$ of x, such that $V_i \cap A \cap \phi_i(X_i) = \emptyset$. Take $V' = \bigcap_{i \in \Lambda} V_i$. Then V' is an open neighborhoodof x in X and $V' \cap A = \emptyset$. Therefore, A is closed in X. □

Corollary 11.4. *Let $\{X, \phi_i\}$ be a limit space for the CIS $\{X_i, Y_i, f_i\}$ in which each X_i is a compact space. If X is Hausdorff and $\{\phi_i(X_i)\}_i$ is a locally finite cover of X, then $\{X, \phi_i\}$ is a fundamental limit space.*

Proof. Each $\phi_i(X_i)$ is a compact subset of the Hasdorff space X. Therefore, each $\phi_i(X_i)$ is closed in X. The result follows from previous theorem. □

Corollary 11.5. *Let $\{X, \phi_i\}$ be a limit space for the finitely semicomponible CIS $\{X_i, Y_i, f_i\}$. Then, $\{X, \phi_i\}$ is a fundamental limit space if and only if the collection $\{\phi_i(X_i)\}_i$ is a locally finite closed cover of X.*

Proof. Poposition 3.2 and Theorems 11.1 and 11.3. □

Let $f : Z \to W$ be a continuous map between topological spaces. We say that f is a **perfect map** if it is closed, surjective and, for each $w \in W$, the subset $f^{-1}(w) \subset Z$ is compact. (See [4]).

Let $\mathfrak{P}$ be a property of topological spaces. We say that $\mathfrak{P}$ is a **perfect property** if always that $\mathfrak{P}$ is true for a space Z and there is a perfect map $f : Z \to W$, we have $\mathfrak{P}$ true for W. Again, we say that a property $\mathfrak{P}$ is **countable-perfect** if $\mathfrak{P}$ is perfect and always that $\mathfrak{P}$ is true for a countable collection of spaces $\{Z_n\}_n$, we have $\mathfrak{P}$ true for the coproduct $\coprod_{n=0}^{\infty} Z_n$. We say that $\mathfrak{P}$ is **finite-perfect** if the previous sentence is true for finite collections $\{Z_n\}_{n=0}^{n_0}$ of topological spaces. It is obvious that every countable-perfect property is also a finite-perfect property. The reciprocal is not true. It is also obvious that every perfect property is a topological invariant.

Example 11.6. *The follows one are examples of countable-prefect properties: Hausdorff axiom, regularity, normality, local compactness, second axiom of enumerability and Lindelöf axiom. The compactness is a finite-perfect property which is not countable-perfect. (For details see* [4]*).*

Theorem 11.7. *Let $\{X, \phi_i\}$ be a fundamental limit space for the finitely semicomponible CIS $\{X_i, Y_i, f_i\}$, in which each X_i has the countable-perfect property $\mathfrak{P}$. Then X has $\mathfrak{P}$.*

Proof. Let $\{X, \phi_i\}$ be a fundamental limit space for $\{X_i, Y_i, f_i\}$. By the Theorems 8.2 and 3.5, there is a unique homeomorphism $\beta : \widetilde{X} \to X$ such that $\phi_i = \beta \circ \widetilde{\varphi}$, for all $i \in \mathbb{N}$. Then, simply to prove that $\widetilde{X}$ has the property $\mathfrak{P}$, where, remember, $\widetilde{X} = (\coprod X_i)/\sim$ is the quotient space constructed in Section 8 (Remember the Remark 8.3).

Consider the quotient map $\rho : \coprod X_i \to \widetilde{X}$. It is continuous and surjective. Moreover, since the CIS $\{X_i, Y_i, f_i\}$ is finitely semicomponible, it is obvious that for $x \in \widetilde{X}$ we have that $\rho^{-1}(x)$ is a finite subset, and so a compact subset, of $\coprod X_i$. Therefore, simply to prove that ρ is a closed map, since this is enough to conclude that ρ is a perfect map and, therefore, the truth of the theorem.

Let $E \subset \coprod X_i$ be an arbitrary closed subset of $\coprod X_i$. We need to prove that $\rho(E)$ is closed in $\widetilde{X}$, that is, that $\rho^{-1}(\rho(E)) \cap X_i$ is closed in X_i for each $i \in \mathbb{N}$. But note that

$$\rho^{-1}(\rho(E)) \cap X_i = (E \cap X_i) \cup \bigcup_{j=0}^{i-1} f_{j,i-1}(E \cap Y_{j,i-1}) \cup \bigcup_{j=i}^{\infty} f_{i,j}^{-1}(E \cap X_{j+1}),$$

where each term of the total union is closed. Now, since the given CIS is finitely semicomponible, there is on the union $\bigcup_{j=i}^{\infty} f_{i,j}^{-1}(E \cap X_{j+1})$ only a finite nonempty terms. Thus, $\rho^{-1}(\rho(E)) \cap X_i$ can be rewritten as a finite union of closed subsets. Therefore $\rho^{-1}(\rho(E)) \cap X_i$ is closed. □

The quotient map $\rho : \coprod X_i \to \widetilde{X}$ is not closed, in general. To illustrate this fact, we introduce the follows example:

Example 11.8. *Consider the inductive CIS $\{S^n, S^n, f_n\}$ as in the Example 4.3, starting at $n = 1$. Consider the sequence of real numbers $(a_n)_n$, where $a_n = 1/n$, $n \geq 1$. Let $A = \{a_n\}_{n\geq 2}$ be the set of points of the sequence $(a_n)_n$ starting at $n = 2$. Then, the image of A by the map $\gamma : [0, 1] \to S^1$ given by $\gamma(t) = (\cos t, \sin t)$ is a sequence $(b_n)_{n\geq 2}$ in S^1 such that the point $b = (1, 0) \in S^1$ is not in $\gamma(A)$ and $(b_n)_n$ converge to b. It follows that the subset $B = \gamma(A)$ of S^1 is not closed in S^1. Now, for each $n \geq 2$, let E^n be the closed $(n-1)$-dimensional half-sphere imbedded as the meridian into S^n going by point $f_{1,n-1}(b_n)$. It is easy to see that E^n is closed in S^n for each $n \geq 2$. Let $E = \bigsqcup_{n=2}^{\infty} E^n$ be the disjoint union of the closed half-spheres E_n. Then, for each $n \geq 2$, $E \cap S^n = E^n$, and $E \cap S^1 = \emptyset$. Thus, E is a closed subset of coproduct space $\coprod_{n=1}^{\infty} S^n$. However, $\rho^{-1}(\rho(E)) \cap S^1 = B$ is not closed in S^1. Hence $\rho(E)$ is not closed in the sphere S^∞. Therefore, the projection $\rho : \coprod S^n \to (\coprod S^n)/\sim\, \cong S^\infty$ is not a closed map.*

Now, we demonstrate the result of the previous theorem in the case of stationary CIS's. In this case the result is stronger, and applies to properties finitely perfect. We started with the following preliminary result, whose proof is obvious and therefore will be omitted (left to the reader).

Lemma 11.9. *Let $\{X, \phi_i\}$ be a fundamental limit space for the stationary CIS $\{X_i, Y_i, f_i\}$. Suppose that this CIS park in the index $n_0 \in \mathbb{N}$. Then $\phi_i = \phi_{n_0}$, for all $i \geq n_0$, and $X \cong \cup_{i=0}^{n_0} \phi_i(X_i)$. Moreover, the composition*

$$\rho_{n_0} : \coprod_{i=0}^{n_0} X_i \xrightarrow{inc.} \coprod_{i=0}^{\infty} X_i \xrightarrow{\rho} \widetilde{X}$$

is a continuous surjection, where inc. indicates the natural inclusion.

Theorem 11.10. *Let $\{X, \phi_i\}$ be a fundamental limit spaces for the stationary CIS $\{X_i, Y_i, f_i\}$, in which each X_i has the finite-perfect property $\mathfrak{P}$. Then X has $\mathfrak{P}$.*

Proof. As in the Theorem 11.7, simply to prove that $\widetilde{X} = (\coprod X_i)/\sim$ has $\mathfrak{P}$.

Suppose that the CIS $\{X_i, Y_i, f_i\}$ parks in the index $n_0 \in \mathbb{N}$. By the previous lemma, the map $\rho_{n_0} : \coprod_{i=0}^{n_0} X_i \to \widetilde{X}$ is continuous and surjective. Thus, simply to prove that ρ_{n_0} is a perfect map. In order to prove this, it rests only to prove that ρ_{n_0} is a closed map and $\rho_{n_0}^{-1}(x)$ is a compact subset of $\coprod_{i=0}^{n_0} X_i$, for each $x \in \widetilde{X}$. This latter fact is trivial, since each subset $\rho_{n_0}^{-1}(x)$ is finite.

In order to prove that ρ_{n_0} is a closed map, let E be an arbitrary closed subset of $\coprod_{i=0}^{n_0} X_i$. We need to prove that $\rho^{-1}(\rho_{n_0}(E)) \cap X_i$ is closed in X_i for each $i \in \mathbb{N}$. But note that, as before,

$$\rho^{-1}(\rho_{n_0}(E)) \cap X_i = (E \cap X_i) \cup \bigcup_{j=0}^{i-1} f_{j,i-1}(E \cap Y_{j,i-1}) \cup \bigcup_{j=i}^{\infty} f_{i,j}^{-1}(E \cap X_{j+1}),$$

where each term of this union is closed. Now, since $E \subset \coprod_{i=0}^{n_0} X_i$, we have $E \cap X_{j+1} = \emptyset$ for all $j \geq n_0$. Thus, the subsets $f_{i,j}^{-1}(E \cap X_{j+1})$ which are in the last part of the union are empty for all $j \geq n_0$. Hence, $\rho^{-1}(\rho_{n_0}(E)) \cap X_i$ is a finite union of closed subsets. Therefore, $\rho^{-1}(\rho_{n_0}(E)) \cap X_i$ is closed. $\square$

References

[1] M. J. Greenberg, *Lectures on Algebraic Topology*, Northeastern University, W. A. Benjamin, Inc. New York, 1967.

[2] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[3] J. J. Hotman, *An introduction to homological algebra*, Acaademic Press, Inc., 1979.

[4] J. R. Munkres, *Topology, a first a course*, Englewood Cliffs; London: Prentice-Hall, 1975.

[5] E. H. Spanier, *Algebraic Topology*, Springer-Verlag New York, Inc. 1966.

[6] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag New York, Inc. 1978.